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HOOK-CONTENT FORMULAE FOR SYMPLECTIC AND ORTHOGONAL TABLEAUX

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Abstract. By considering the specialisation $s_\lambda(1, q, q^2, \ldots, q^{n-1})$ of the Schur function, Stanley was able to describe a formula for the number of semistandard Young tableaux of shape λ in terms of two properties of the boxes in the diagram for λ. Using specialisations of symplectic and orthogonal Schur functions, we derive corresponding formulae, first given by El Samra and King, for the number of semistandard symplectic and orthogonal λ-tableaux.

1. Introduction

To each partition λ with at most n parts there corresponds an irreducible polynomial representation of the general linear group $\mathrm{GL}(n)$ over the field of complex numbers. Indeed, this representation has a basis indexed by semistandard Young tableaux of shape λ with entries from $\{1, 2, \ldots, n\}$. The number of semistandard λ-tableaux is therefore equal to the dimension of the representation and this is given by Weyl's dimension formula [16].

However, a more combinatorial description of the number of semistandard λ-tableaux was derived by Stanley [12] using Weyl's character formula. The character corresponding to the partition λ is the Schur function $s_\lambda(x_1, \ldots, x_n)$ and Stanley showed that its specialisation $s_\lambda(1, q, q^2, \ldots, q^{n-1})$ could be expressed as a product involving the hook lengths and contents of the boxes in the diagram for λ. This provides a generating function for the semistandard λ-tableaux with entries in $\{1, 2, \ldots, n\}$ and, in particular, taking $q = 1$ yields a formula the number of such tableaux.

A similar situation exists for the classical groups $\mathrm{Sp}(2n)$ and $\mathrm{O}(m)$ over the complex numbers. Semistandard symplectic and orthogonal λ-tableaux have been introduced by various authors (see [2], [7] and [8], [10], [15]) and it is known that they index bases for the irreducible polynomial representations associated to λ for $\mathrm{Sp}(2n)$ [1] and $\mathrm{O}(m)$ [8], respectively. El Samra and King [5] were then able to manipulate Weyl's dimension formula to produce formulae for the number of semistandard symplectic and odd orthogonal λ-tableaux in terms of hook lengths and contents.

The aim of this paper is to adapt Stanley's approach using Weyl's character formula to these cases. We obtain expressions for the specialisations $sp_\lambda(q, q^3, q^5, \ldots, q^{2n-1})$, $o_\lambda(q^2, q^4, q^6, \ldots, q^{2n})$ and $o_\lambda(q, q^3, q^5, \ldots, q^{2n-1})$ of the symplectic, odd orthogonal

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and even orthogonal Schur functions. These give generating functions for the semistandard symplectic and orthogonal tableaux of shape λ with entries in the sets $\{1,\overline{1},2,\overline{2},\ldots,n,\overline{n}\}$ or $\{1,\overline{1},2,\overline{2},\ldots,n,\overline{n},\infty\}$ as appropriate and we recover the formulae from [5] as a special case by setting $q=1$. The specialisations in the symplectic and odd orthogonal cases have previously been studied by Koike [9]. However, our approach is more direct, is closer to the approach taken by Stanley, and the result we obtain is different in the even orthogonal case. Further, Koike was not concerned with the associated tableaux but instead the quantum dimensions of the irreducible modules.

We begin with a preliminary section which recalls the basic combinatorics of Young tableaux with the following section describing the symplectic and orthogonal tableaux that we will be considering. In the remaining sections we derive the generating functions for the symplectic and orthogonal cases.

2. Young tableaux

A *partition* of a positive integer r is a k-tuple $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ of weakly decreasing non-negative integers such that $\sum_{i=1}^k \lambda_i = r$. The non-zero λ_i in the k-tuple are called the *parts* of λ. The *Young diagram* of shape λ is the subset of $\mathbb{Z}^2$ defined by

$$[\lambda]=\{(i,j)\mid i,j\in\mathbb{N},\ 1\le i\le k,\ 1\le j\le \lambda_i\}.$$

This is represented in the plane by arranging r boxes in k left-justified rows with the ith row containing λ_i boxes. The *conjugate* of λ is the partition $\lambda^t=(\lambda_1^t,\lambda_2^t,\ldots,\lambda_s^t)$ where λ_i^t is the number of boxes in the ith column of the Young diagram of shape λ. One obtains a λ-tableau by filling $[\lambda]$ with entries from a set $\{1,2,\ldots,n\}$ where n is a positive integer. A λ-tableau is *semistandard* if the entries in each row are weakly increasing from left to right and the entries in each column are strictly increasing from top to bottom.

Each box in $[\lambda]$ has an associated *hook* which consists of that box, all boxes to the right of it in that row and all boxes below it in that column. The *hook length* of the box is then the number of boxes in its hook. Specifically, for $(i,j)\in[\lambda]$ we have $h(i,j)=\lambda_i+\lambda_j^t-i-j+1$.

Using the Schur function $s_\lambda(x_1,x_2,\ldots,x_n)$, Stanley [12] obtained a formula for the number $d(\lambda,n)$ of semistandard λ-tableaux with entries in the set $\{1,2,\ldots,n\}$:

$$d(\lambda,n)=\prod_{(i,j)\in[\lambda]}\frac{n+c(i,j)}{h(i,j)}$$

where $c(i,j)=j-i$ is the *content* of the (i,j)th box. More generally, he proved that for an indeterminate q

$$s_\lambda(1,q,\ldots,q^{n-1})=q^{b(\lambda)}\prod_{(i,j)\in[\lambda]}\frac{[n+c(i,j)]}{[h(i,j)]} \tag{1}$$

where $b(\lambda)=\sum_{i=1}^k(i-1)\lambda_i$ and $[i]=q^i-1$. If we let $|T|$ denote the sum of the entries in the tableau T, then the coefficient of q^i in $s_\lambda(1,q,\ldots,q^{n-1})$ is the number of semistandard λ-tableaux with entries in $\{1,2,\ldots,n\}$ which have $|T|=i+r$.

Consequently, (1) can be interpreted as providing a generating function for such tableaux.

3. Symplectic and orthogonal tableaux

Throughout, fix positive integers r and n and let λ be a partition of r into at most n parts. The symplectic tableaux that we will be considering were defined by King [7] and require the $2n$ symbols $1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$. However, we may equivalently have taken the symplectic tableaux of De Concini [2] since there is a weight preserving bijection between these and the King tableaux (see [11]).

Definition 1. A *semistandard symplectic tableau* of shape λ is a semistandard λ-tableau T with entries from the set $\{1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}\}$ such that the entries in the ith row of T are greater than or equal to i for each i.

Given a symplectic tableau T, let $a_i(T)$ be the number of appearances of the symbol i as an entry in T. The *weight* of T is the monomial in the variables $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$ defined by

$$\mathrm{wt}(T) = \prod_{i=1}^{n} x_i^{a_i(T) - a_{\overline{i}}(T)}. \tag{2}$$

The *symplectic Schur function* corresponding to λ is

$$sp_\lambda(x_1, x_2, \ldots, x_n) = \sum_T \mathrm{wt}(T),$$

where the sum runs over all semistandard symplectic λ-tableaux, and it is the character of the irreducible polynomial $\mathrm{Sp}(2n)$-module with highest weight λ [6].

Example. Let $\lambda = (1,1)$ and $n = 2$. The semistandard symplectic λ-tableaux are

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline 1 \\ \hline \overline{2} \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline \overline{1} \\ \hline 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline \overline{1} \\ \hline \overline{2} \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array}$$

so the corresponding symplectic Schur function is

$$sp_\lambda(x_1, x_2) = x_1 x_2 + x_1 x_2^{-1} + x_1^{-1} x_2 + x_1^{-1} x_2^{-1} + 1.$$

The orthogonal tableaux that we will consider are those introduced by King and Welsh [8], but there is a weight preserving bijection (see [8]) between these tableaux and the various orthogonal tableaux given by Proctor in [10, Section 6]. Further, in the odd case there is a weight preserving bijection (see [3]) between Proctor's tableaux and the odd orthogonal tableaux of Sundaram [15].

For the even orthogonal tableaux we use the same set of symbols as for the symplectic tableaux. However, for the odd tableaux we will need the $2n+1$ symbols $1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n} < \infty$ where we take $\overline{\infty} = \infty$. In the following definition α_i and β_i denote the number of entries that are at most $\overline{i}$ in the first and second columns of T, respectively, and $T_{i,j}$ denotes the entry in the (i,j)th box of T.

Definition 2. A *semistandard even orthogonal* or *odd orthogonal tableau* of shape λ is a semistandard λ-tableau T with entries from the set $\{1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}\}$ or $\{1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}, \infty\}$, respectively, such that for each $1 \leq i \leq n$

(i) $\alpha_i + \beta_i \leq 2i$;
(ii) if $\alpha_i + \beta_i = 2i$ with $\alpha_i > \beta_i$ and $T_{\alpha_i,1} = \overline{i}$ and $T_{\beta_i,2} = i$ then $T_{\alpha_i-1,1} = i$; and
(iii) if $\alpha_i + \beta_i = 2i$ with $\alpha_i = \beta_i = i$ and $T_{\alpha_i,1} = i$ and $T_{\alpha_i,j} = \overline{i}$ then $T_{\alpha_i-1,j} = i$.

Let $\mathrm{wt}(T)$ denote the weight of the tableau T as given in (2). The *even orthogonal* and *odd orthogonal Schur functions* corresponding to λ are defined to be

$$o_\lambda(x_1, x_2, \ldots, x_n) = \sum_T \mathrm{wt}(T),$$

where the sum runs over the semistandard even orthogonal and odd orthogonal λ-tableaux respectively. These are then the characters for the irreducible polynomial $\mathrm{O}(2n)$ and $\mathrm{O}(2n+1)$-modules of highest weight λ [10].

Example. Let $\lambda = (1,1)$ and $n = 2$. The semistandard even orthogonal λ-tableaux are

$$\begin{array}{|c|}\hline 1\\\hline \overline{1}\\\hline\end{array},\quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\quad \begin{array}{|c|}\hline 1\\\hline \overline{2}\\\hline\end{array},\quad \begin{array}{|c|}\hline \overline{1}\\\hline 2\\\hline\end{array},\quad \begin{array}{|c|}\hline \overline{1}\\\hline \overline{2}\\\hline\end{array},\quad \begin{array}{|c|}\hline 2\\\hline \overline{2}\\\hline\end{array}.$$

Here we see that there are two tableaux of weight 0 so the corresponding even orthogonal Schur function is

$$o_\lambda(x_1, x_2) = x_1x_2 + x_1x_2^{-1} + x_1^{-1}x_2 + x_1^{-1}x_2^{-1} + 2.$$

The odd orthogonal λ-tableaux are allowed the additional symbol ∞ so we have

$$\begin{array}{|c|}\hline 1\\\hline \overline{1}\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\\hline \overline{2}\\\hline\end{array},\ \begin{array}{|c|}\hline \overline{1}\\\hline 2\\\hline\end{array},\ \begin{array}{|c|}\hline \overline{1}\\\hline \overline{2}\\\hline\end{array},\ \begin{array}{|c|}\hline 2\\\hline \overline{2}\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\\hline \infty\\\hline\end{array},\ \begin{array}{|c|}\hline \overline{1}\\\hline \infty\\\hline\end{array},\ \begin{array}{|c|}\hline 2\\\hline \infty\\\hline\end{array},\ \begin{array}{|c|}\hline \overline{2}\\\hline \infty\\\hline\end{array}$$

and the corresponding odd orthogonal Schur function is

$$o_\lambda(x_1, x_2) = x_1x_2 + x_1x_2^{-1} + x_1^{-1}x_2 + x_1^{-1}x_2^{-2} + x_1 + x_1^{-1} + x_2 + x_2^{-1} + 2.$$

The irreducible $\mathrm{O}(2n+1)$-module of highest weight λ remains irreducible on restriction to the special orthogonal group $\mathrm{SO}(2n+1)$ and the same is true for the irreducible $\mathrm{O}(2n)$-module when λ has strictly fewer than n parts. However, if λ has exactly n parts then the restriction decomposes as the direct sum of the irreducible $\mathrm{SO}(2n)$-modules of highest weights $\lambda^+ = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n)$ and $\lambda^- = (\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n)$. The subsets of even orthogonal tableaux corresponding to the two $\mathrm{SO}(2n)$-modules are described by the following restatement of [8, Definition 4.15].

Definition 3. Let λ be a partition with exactly n parts. An even orthogonal tableau T of shape λ is *positive* if

(i) the first entry in the ith column of T is either i or $\overline{i}$ for each $1 \leq i \leq n$ and the number of rows starting with a symbol in $\{1, 2, \ldots, n\}$ is even; or
(ii) the first entry in the ith column of T is either i or $\overline{i}$ for each $1 \leq i \leq j$, but the jth row starts with a symbol greater than $\overline{j}$ for some $j < n$.

Similarly, T is *negative* if it satisfies either (i) with even replaced by odd or (ii) exactly as above. In particular, a tableau may be both positive and negative, or neither.

Example. Let $\lambda = (1,1)$ and $n = 2$. The positive even orthogonal λ-tableaux are

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline \overline{1} \\ \hline \overline{2} \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array}$$

while the negative even orthogonal λ-tableaux are

$$\begin{array}{|c|}\hline 1 \\ \hline \overline{2} \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline \overline{1} \\ \hline 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array}\,.$$

4. Generating function for semistandard symplectic tableaux

Our aim is to produce an analogue of (1) for the symplectic Schur function and we start with the determinantal formula [4, Equation 24.18]

$$sp_\lambda(x_1, x_2, \ldots, x_n) = \frac{|x_j^{\lambda_i+n-i+1} - x_j^{-\lambda_i-n+i-1}|_{i,j=1}^n}{|x_j^{n-i+1} - x_j^{-n+i-1}|_{i,j=1}^n}. \tag{3}$$

Let q be an indeterminate and for a positive integer i define $\langle i \rangle = q^i - q^{-i}$ with $\langle i \rangle! = \langle 1 \rangle \langle 2 \rangle \cdots \langle i-1 \rangle \langle i \rangle$. When necessary, we will also set $\langle 0 \rangle! = \langle 0 \rangle = 1$. The following result is a generalisation of [4, Exercise 24.20].

Lemma 4. *Let λ be a partition with at most n parts and set $\mu_i = \lambda_i + n - i$. Then*

$$sp_\lambda(q, q^3, q^5, \ldots, q^{2n-1}) = \frac{\prod_{i=1}^n \langle \mu_i + 1 \rangle \prod_{1 \le i < j \le n} \langle \mu_i - \mu_j \rangle \langle \mu_i + \mu_j + 2 \rangle}{\prod_{i=1}^n \langle 2i-1 \rangle!}. \tag{4}$$

Proof. Setting $x_j = q^{2j-1}$ in (3) we have

$$sp_\lambda(q, q^3, q^5, \ldots, q^{2n-1}) = \frac{|q^{(2j-1)(\mu_i+1)} - q^{-(2j-1)(\mu_i+1)}|_{i,j=1}^n}{|q^{(2j-1)(n-i+1)} - q^{-(2j-1)(n-i+1)}|_{i,j=1}^n}. \tag{5}$$

Let d and d' denote the denominator and numerator of (5) respectively. Elementary row operations allows us to rewrite the denominator as

$$d = (-1)^{n(n-1)/2} \prod_{j=1}^n (q^{2j-1} - q^{1-2j}) |(q^{2j-1} + q^{1-2j})^{i-1}|_{i,j=1}^n.$$

The determinant in this expression is the determinant of the transpose of a Vandermonde matrix so we have

$$\begin{aligned} |(q^{2j-1} + q^{1-2j})^{i-1}|_{i,j=1}^n &= \prod_{1 \le i < j \le n} ((q^{2j-1} + q^{1-2j}) - (q^{2i-1} + q^{1-2i})) \\ &= \prod_{1 \le i < j \le n} (q^{j-i} - q^{i-j})(q^{i+j-1} - q^{1-i-j}). \end{aligned}$$

Thus, we obtain

$$d = (-1)^{n(n-1)/2} \prod_{i=1}^n \prod_{k=1}^{2i-1} (q^k - q^{-k}) = (-1)^{n(n-1)/2} \prod_{i=1}^n \langle 2i-1 \rangle!.$$

For the numerator we proceed similarly:

$$d' = \prod_{i=1}^{n}(q^{\mu_i+1} - q^{-\mu_i-1})|(q^{\mu_i+1} + q^{-\mu_i-1})^{2(j-1)}|_{i,j=1}^{n}$$

with

$$\begin{aligned} |(q^{\mu_i+1} + q^{-\mu_i-1})^{2(j-1)}|_{i,j=1}^{n} &= \prod_{1\le i<j\le n}((q^{\mu_j+1} + q^{-\mu_j-1})^2 - (q^{\mu_i+1} + q^{-\mu_i-1})^2) \\ &= \prod_{1\le i<j\le n}(q^{\mu_j-\mu_i} - q^{\mu_i-\mu_j})(q^{\mu_i+\mu_j+2} - q^{-\mu_i-\mu_j-2}). \end{aligned}$$

Hence,

$$d' = (-1)^{n(n-1)/2}\prod_{i=1}^{n}\langle\mu_i+1\rangle\prod_{1\le i<j\le n}\langle\mu_i-\mu_j\rangle\langle\mu_i+\mu_j+2\rangle$$

and the result follows. □

We need to identify the right-hand side of (4) as a suitable product over the boxes in the diagram. Firstly, we consider the contribution from the hook lengths. Although this is identical to the result for $\mathrm{GL}(n)$, for completeness we provide a proof.

Lemma 5. *Let λ be a partition of at most n parts. Then*

$$\prod_{(i,j)\in[\lambda]}\langle h(i,j)\rangle = \frac{\prod_{i=1}^{n}\langle\mu_i\rangle!}{\prod_{1\le i<j\le n}\langle\mu_i-\mu_j\rangle}.$$

Proof. Consider only the ith row of the diagram. The hook lengths strictly decrease as we move from left to right along the row so it is enough to show that for each $1 \le j \le \lambda_i$ we have $h(i,j) \le \mu_i$ but that $h(i,j) \ne \mu_i - \mu_\ell$ for any $i < \ell \le n$. Let $\ell = \lambda_j^t$ so that $h(i,j) = (\lambda_i - i) + (\ell - j) + 1$. Then $\lambda_{\ell+1} < j < \lambda_\ell + 1$ implies that $\mu_i - \mu_\ell < h(i,j) < \mu_i - \mu_{\ell+1}$ where this gives $\mu_i - \mu_n < h(i,j) \le \mu_i$ for $\ell = n$. □

In the case of $\mathrm{GL}(n)$ the content of a box is defined independently of the partition. For symplectic tableaux, however, we take (see [14])

$$r_\lambda(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 2 & \text{if } i > j, \\ i + j - \lambda_i^t - \lambda_j^t & \text{if } i \le j. \end{cases}$$

Lemma 6. *Let λ be a partition of at most n parts. Then*

$$\prod_{(i,j)\in[\lambda]}\langle 2n + r_\lambda(i,j)\rangle = \frac{\prod_{i=1}^{n}\langle\mu_i+1\rangle!\prod_{1\le i<j\le n}\langle\mu_i+\mu_j+2\rangle}{\prod_{i=1}^{n}\langle 2i-1\rangle!}. \tag{6}$$

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ so that $k = \lambda_1^t$ and consider $\lambda' = (\lambda_2 - 1, \ldots, \lambda_k - 1)$, the partition obtained by removing the hook with corner $(1,1)$ from the diagram for λ. The content of the (i,j)th box in $[\lambda']$ is then $r_{\lambda'}(i,j) = r_\lambda(i+1, j+1)$ which means that we may calculate (6) by induction on the number of parts of λ. We begin by examining the entries of the boxes lying in the $(1,1)$-hook of λ.

Recall that we defined $\mu_i = \lambda_i + n - i$ for each $1 \le i \le n$. It is clear that for $1 < i \le k$ we have $2n + r_\lambda(i,1) = \mu_1 + \mu_i + 2$. Further, for $1 \le j \le \lambda_1$ we may write

$2n + r_\lambda(1, j) = (n - k) + (\mu_1 - h(1, j)) + 2$ where $h(1, j)$ is the corresponding hook length in $[\lambda]$. From the proof of Lemma 5 we know that as we run along the first row $h(1, j)$ will take on the values 1 up to μ_1 excluding those of the form $\mu_1 - \mu_\ell$ for $1 < \ell \le n$. Thus

$$\prod_{j=1}^{\lambda_1} \langle 2n + r_\lambda(1, j)\rangle = \frac{\langle (n-k) + \mu_1 + 1\rangle!}{\langle (n-k) + 1\rangle \prod_{\ell=2}^{n} \langle (n-k) + \mu_\ell + 2\rangle}.$$

However, $\mu_i = n - i$ for all $k < i \le n$ so we may express $\langle (n - k) + \mu_1 + 1\rangle!$ in the numerator as the product of $\langle \mu_1 + \mu_{k+1} + 2\rangle \cdots \langle \mu_1 + \mu_n + 2\rangle$ and $\langle \mu_1 + 1\rangle!$. Consequently, the product over all the boxes in the hook is

$$\prod_{(i,j)\in[(\lambda_1,1^{k-1})]} \langle 2n + r_\lambda(i, j)\rangle = \frac{\langle \mu_1 + 1\rangle! \prod_{j=2}^{n} \langle \mu_1 + \mu_j + 2\rangle}{\langle (n-k) + 1\rangle! \prod_{j=2}^{n} \langle (n-k) + \mu_j + 2\rangle}. \tag{7}$$

To prove the base case suppose that the diagram for λ is a single hook; that is, $\lambda = (\lambda_1, 1^{k-1})$. This gives $\mu_i = n - i + 1$ for $1 < i \le k$ and $n - i$ for $i > k$. In particular,

$$\langle \mu_i + 1\rangle! \prod_{j=i+1}^{n} \langle \mu_i + \mu_j + 2\rangle = \begin{cases} \langle 2(n-i+2) - 1\rangle! / \langle (n-k) + \mu_i + 2\rangle & \text{if } 1 < i \le k; \\ \langle 2(n-i+1) - 1\rangle! & \text{if } k < i \le n \end{cases}$$

so

$$\prod_{i=2}^{n} \langle \mu_i + 1\rangle! \prod_{2\le i<j\le n} \langle \mu_i + \mu_j + 2\rangle = \frac{\prod_{i=1}^{n} \langle 2i - 1\rangle!}{\langle 2(n-k) + 1\rangle! \prod_{i=2}^{k} \langle (n-k) + \mu_i + 2\rangle}.$$

Moreover, we can replace the factorial $\langle 2(n - k) + 1\rangle!$ in the denominator by the product of $\langle (n - k) + \mu_{k+1} + 2\rangle \cdots \langle (n - k) + \mu_n + 2\rangle$ and $\langle (n - k) + 1\rangle!$. Hence we find that (7) is equivalent to (6) in this case.

Now suppose that $[\lambda]$ is more than a single hook and let $\lambda' = (\lambda_2 - 1, \ldots, \lambda_k - 1)$ as above. By induction we know that

$$\prod_{(i,j)\in[\lambda']} \langle 2n + r_{\lambda'}(i, j)\rangle = \frac{\prod_{i=1}^{n} \langle \mu'_i + 1\rangle! \prod_{1\le i<j\le n} \langle \mu'_i + \mu'_j + 2\rangle}{\prod_{i=1}^{n} \langle 2i - 1\rangle!}$$

where $\mu'_i = \lambda'_i + n - i$. Here we see that $\mu'_i = \mu_{i+1}$ for $1 \le i < k$ and μ_i for $k < i \le n$ with $\mu'_k = n - k$. We may therefore reexpress this as

$$\frac{\prod_{(i,j)\in[\lambda']} \langle 2n + r_\lambda(i + 1, j + 1)\rangle}{\langle (n-k) + 1\rangle! \prod_{j=2}^{n} \langle (n-k) + \mu_j + 2\rangle} = \frac{\prod_{i=2}^{n} \langle \mu_i + 1\rangle! \prod_{2\le i<j\le n} \langle \mu_i + \mu_j + 2\rangle}{\prod_{i=1}^{n} \langle 2i - 1\rangle!}.$$

Combining with (7) we obtain (6) and we are done. □

Remark 7. In [13] the formula for the product of the hook lengths is derived by manipulating the diagram for λ. We add $k-i$ boxes to the ith row of the diagram, fill the row with the numbers 1 up to $\mu_i-(n-k)$ starting from the right, and remove the columns $1 + \mu_j$ for $1 < j \le n$. The boxes remaining form the diagram for λ with the hook lengths in the appropriate places while the boxes removed are precisely those

containing $\mu_i - \mu_j$ for $1 \le i < j \le k$. For example, when $\lambda = (7,5,4,1)$ and $n = 4$ we have $\mu = (10,7,5,1)$ and obtain

$$\begin{array}{cccccccccc} 10 & \mathbf{9} & 8 & 7 & 6 & \mathbf{5} & 4 & \mathbf{3} & 2 & 1 \\ 7 & \mathbf{6} & 5 & 4 & 3 & \mathbf{2} & 1 & & & \\ 5 & \mathbf{4} & 3 & 2 & 1 & & & & & \\ 1 & & & & & & & & & \end{array} \quad \longrightarrow \quad \begin{array}{ccccccc} 10 & 8 & 7 & 6 & 4 & 2 & 1 \\ 7 & 5 & 4 & 3 & 1 & & \\ 5 & 3 & 2 & 1 & & & \\ 1 & & & & & & \end{array}$$

A similar method can be used to derive (7), the formula for the product of the contents in the $(1,1)$-hook. We add $k-1$ boxes to the arm of the hook and label the boxes in the following way: the leg of the hook, excluding $(1,1)$, is filled with the numbers $\mu_1 + \mu_j + 2$ for $1 < j \le k$ starting from the top and the arm of the hook, including $(1,1)$, with $2(n-k)+2$ up to $\mu_1 + (n-k)+1$ starting from the left. We then remove the boxes in the arm at positions $\mu_j - (n-k)+1$ for $1 < j \le k$. The eliminated boxes contain $\mu_j + (n-k)+2$ for $1 < j \le k$ and the remaining boxes the values of $2n + r_\lambda(i,j)$ for the hook. We therefore have

$$\prod_{(i,j)\in[(\lambda_1,1^{k-1})]} \langle 2n + r_\lambda(i,j)\rangle = \frac{\langle \mu_1 + (n-k)+1\rangle!}{\langle 2(n-k)+1\rangle! \prod_{j=1}^{k} \langle (n-k)+\mu_j+2\rangle}.$$

and note that $\langle \mu_1 + (n-k)+1\rangle! = \langle \mu_1+1\rangle!\langle \mu_1 + \mu_{k+1}+2\rangle \cdots \langle \mu_1+\mu_n+2\rangle$ while $\langle 2(n-k)+1\rangle! = \langle (n-k)+1\rangle!\langle (n-k)+\mu_{k+1}+2\rangle \cdots \langle (n-k)+\mu_n+2\rangle$.

For example, $\lambda = (7,5,4,1)$ with $n=4$ and $\mu = (10,7,5,1)$ gives for the first hook

$$\begin{array}{cccccccccc} 2 & \mathbf{3} & 4 & 5 & 6 & \mathbf{7} & 8 & \mathbf{9} & 10 & 11 \\ 19 & & & & & & & & & \\ 17 & & & & & & & & & \\ 13 & & & & & & & & & \end{array} \quad \longrightarrow \quad \begin{array}{ccccccc} 2 & 4 & 5 & 6 & 8 & 10 & 11 \\ 19 & & & & & & \\ 17 & & & & & & \\ 13 & & & & & & \end{array}$$

To continue, we consider the partition $\lambda' = (4,3)$ where we still have $n=4$ but now $k=2$. Consequently, $\mu' = (7,5,1,0)$ and we obtain for the second hook

$$\begin{array}{ccccc} 6 & 7 & 8 & \mathbf{9} & 10 \\ 14 & & & & \end{array} \quad \longrightarrow \quad \begin{array}{cccc} 6 & 7 & 8 & 10 \\ 14 & & & \end{array}$$

Finally, for the third hook we use $\lambda'' = (2)$ with $\mu'' = (5,2,1,0)$ which simply produces

$$\begin{array}{cc} 8 & 9 \end{array}$$

Combining yields the complete diagram:

$$\begin{array}{ccccccc} 2 & 4 & 5 & 6 & 8 & 10 & 11 \\ 19 & 6 & 7 & 8 & 10 & & \\ 17 & 14 & 8 & 9 & & & \\ 13 & & & & & & \end{array}$$

Theorem 8. *Let λ be a partition with at most n parts. Then*

$$sp_\lambda(q, q^3, \ldots, q^{2n-1}) = \prod_{(i,j)\in[\lambda]} \frac{\langle 2n + r_\lambda(i,j)\rangle}{\langle h(i,j)\rangle}.$$

Proof. From the previous three Lemmas we see that

$$\begin{aligned} sp_\lambda(q,q^3,\ldots,q^{2n-1}) &= \frac{\prod_{i=1}^n\langle\mu_i+1\rangle\prod_{1\le i<j\le n}\langle\mu_i-\mu_j\rangle\langle\mu_i+\mu_j+2\rangle}{\prod_{j=1}^n\langle 2j-1\rangle!}\frac{\prod_{i=1}^n\langle\mu_i\rangle!}{\prod_{i=1}^n\langle\mu_i\rangle!} \\ &= \frac{\prod_{i=1}^n\langle\mu_i+1\rangle!\prod_{1\le i<j\le n}\langle\mu_i+\mu_j+2\rangle}{\prod_{i=1}^n\langle 2i-1\rangle!}\frac{\prod_{1\le i<j\le n}\langle\mu_i-\mu_j\rangle}{\prod_{i=1}^n\langle\mu_i\rangle!} \\ &= \prod_{(i,j)\in[\lambda]}\langle 2n+r_\lambda(i,j)\rangle\prod_{(i,j)\in[\lambda]}\frac{1}{\langle h(i,j)\rangle} \end{aligned}$$

as required. □

Define $|T|$ to be the sum of the entries of the symplectic tableau T where the symbol $\bar{i}$ is counted as $-i$. Let $r(T)=r_+(T)-r_-(T)$ where $r_+(T)$ and $r_-(T)$ are the number of boxes of T containing a symbol from the sets $\{1,\ldots,n\}$ and $\{\overline{1},\ldots,\overline{n}\}$ respectively. Then

$$sp_\lambda(q,q^3,q^5,\ldots,q^{2n-1})=\sum_T q^{2|T|-r(T)},$$

where T runs over the semistandard symplectic λ-tableaux, so Theorem 8 provides a generating function for such tableaux. As a special case, setting $q=1$ recovers King and El-Samra's expression [5] for the dimension of the irreducible polynomial $\mathrm{Sp}(2n)$-module with highest weight λ.

Corollary 9. *The number of semistandard symplectic λ-tableaux with entries in the set $\{1,\overline{1},2,\overline{1},\ldots,n,\overline{n}\}$ is*

$$d_{sp}(\lambda,2n)=\prod_{(i,j)\in[\lambda]}\frac{2n+r_\lambda(i,j)}{h(i,j)}.$$

5. Generating function for semistandard odd orthogonal tableaux

Although our approach for the orthogonal tableaux will be identical to that for the symplectic tableaux, we need to consider the odd orthogonal and even orthogonal cases separately. We begin with the odd orthogonal tableaux where the relevant determinantal formula is [4, Equation 24.28]

$$o_\lambda(x_1,x_2,\ldots,x_n)=\frac{|x_j^{\lambda_i+n-i+1/2}-x_j^{-\lambda_i-n+i-1/2}|_{i,j=1}^n}{|x_j^{n-i+1/2}-x_j^{-n+i-1/2}|_{i,j=1}^n}$$

and we wish to generalise [4, Exercise 24.30].

Lemma 10. *Let λ be a partition with at most n parts and set $\mu_i=\lambda_i+n-i$. Then*

$$o_\lambda(q^2,q^4,\ldots,q^{2n})=\frac{\prod_{i=1}^n\langle 2\mu_i+1\rangle\prod_{1\le i<j\le n}\langle\mu_i-\mu_j\rangle\langle\mu_i+\mu_j+1\rangle}{\prod_{i=1}^n\langle 2i-1\rangle!}.$$

Proof. As in the proof of Lemma 4, we apply the determinantal formula to get

$$o_\lambda(q^2, q^4, \ldots, q^{2n}) = \frac{|q^{j(2\mu_i+1)} - q^{-j(2\mu_i+1)}|_{i,j=1}^n}{|q^{j(2n-2i+1)} - q^{-j(2n-2i+1)}|_{i,j=1}^n}$$

and manipulate both the denominator d and numerator d'. Indeed, after transposing and reversing the order of the rows and columns, we see that the denominator in this case is equal to the denominator in (5); that is,

$$d = (-1)^{n(n-1)/2} \prod_{j=1}^n \langle 2j-1\rangle !.$$

Further, the numerator can be expressed as

$$d' = \prod_{i=1}^n (q^{2\mu_i+1} - q^{-2\mu_i-1}) |(q^{2\mu_i+1} - q^{-2\mu_i-1})^{j-1}|_{i,j=1}^n$$

with

$$\begin{aligned} |(q^{2\mu_i+1} - q^{-2\mu_i-1})^{j-1}|_{i,j=1}^n &= \prod_{1\le i<j\le n} ((q^{2\mu_j+1} + q^{-2\mu_j-1}) - (q^{2\mu_i+1} - q^{-2\mu_i-1})) \\ &= \prod_{1\le i<j\le n} (q^{\mu_j-\mu_i} - q^{-\mu_i-\mu_j})(q^{\mu_i+\mu_j+1} - q^{-\mu_i-\mu_j-1}). \end{aligned}$$

Hence,

$$d' = (-1)^{n(n-1)/2} \prod_{i=1}^n \langle 2\mu_i + 1\rangle \prod_{1\le i<j\le n} \langle \mu_i - \mu_j\rangle \langle \mu_i + \mu_j + 1\rangle$$

and we are done. $\square$

For odd orthogonal λ-tableaux, the content of the (i,j)th box is (see [14])

$$r'_\lambda(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j & \text{if } i \ge j, \\ i + j - \lambda_i^t - \lambda_j^t - 2 & \text{if } i < j. \end{cases}$$

Lemma 11. *Let λ be a partition with at most n parts. Then*

$$\prod_{(i,j)\in[\lambda]} \langle 2n+1+r'_\lambda(i,j)\rangle = \frac{\prod_{i=1}^n \langle \mu_i\rangle ! \prod_{1\le i\le j\le n} \langle \mu_i + \mu_j + 1\rangle}{\prod_{i=1}^n \langle 2i-1\rangle !}. \tag{8}$$

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ so that $k = \lambda_1^t$. Again, we proceed by induction on the number of parts in λ and begin by considering only the boxes in the $(1,1)$-hook of λ.

For $1 \le i \le k$ we have $2n+1+r'_\lambda(i,1) = \mu_1 + \mu_i + 1$. Further, for $1 < j \le \lambda_1$ we see that $2n+1+r'_\lambda(1,j) = (n-k) + (\mu_1 - h(1,j)) + 1$. The hook lengths $h(1,j)$ for the first row of $[\lambda]$ take on the values 1 up to μ_1 except those of the form $\mu_1 - \mu_\ell$ for $1 < \ell \le n$. However, we also need to exclude $h(1,1) = \lambda_1 + k - 1$ so we find that

$$\prod_{j=2}^{\lambda_1} \langle 2n+1+r'_\lambda(1,j)\rangle = \frac{\langle (n-k)+\mu_1\rangle !}{\langle n-k\rangle ! \langle 2(n-k)+1\rangle \prod_{\ell=2}^n \langle (n-k)+\mu_\ell+1\rangle}.$$

Replacing $\langle (n-k)+\mu_1\rangle!$ by the product of $\langle \mu_1+\mu_{k+1}+1\rangle\cdots\langle\mu_1+\mu_n+1\rangle$ and $\langle\mu_1\rangle!$ we obtain
(9)

$$\prod_{(i,j)\in[(\lambda_1,1^{k-1})]}\langle 2n+1+r'_\lambda(i,j)\rangle = \frac{\langle\mu_1\rangle!\prod_{j=1}^{n}\langle\mu_1+\mu_j\rangle}{\langle n-k\rangle!\langle 2(n-k)+1\rangle\prod_{j=2}^{n}\langle (n-k)+\mu_j+1\rangle}.$$

When $\lambda=(\lambda_1,1^{k-1})$ is a single hook, we find that

$$\langle\mu_i\rangle!\prod_{i=j}^{n}\langle\mu_i+\mu_j+1\rangle = \begin{cases}\langle 2(n-i+2)-1\rangle!/\langle (n-k)+\mu_i+1\rangle & \text{if } 1<i\le k,\\ \langle 2(n-i+1)-1\rangle! & \text{if } k<i\le n.\end{cases}$$

Thus

$$\prod_{i=2}^{n}\langle\mu_i\rangle!\prod_{2\le i\le j\le n}\langle\mu_i+\mu_j+1\rangle = \frac{\prod_{i=1}^{n}\langle 2i-1\rangle!}{\langle n-k\rangle!\langle 2(n-k)+1\rangle\prod_{i=2}^{n}\langle (n-k)+\mu_i+1\rangle}.$$

and (9) is equivalent to (8) in the base case.

In general, let $\lambda'=(\lambda_2-1,\dots,\lambda_k-1)$ and define $\mu'_i=\lambda'_i+n-i$ for each i. As before, $\mu'_i=\mu_{i+1}$ for $1\le i<k$ and μ_i for $k<i\le n$ with $\mu'_k=n-k$. Consequently, by induction

$$\frac{\prod_{(i,j)\in[\lambda']}\langle 2n+1+r'_\lambda(i+1,j+1)\rangle}{\langle n-k\rangle!\langle 2(n-k)+1\rangle\prod_{i=2}^{n}\langle (n-k)+\mu_j+1\rangle} = \frac{\prod_{i=2}^{n}\langle\mu_i\rangle!\prod_{2\le i\le j\le n}\langle\mu_i+\mu_j+1\rangle}{\prod_{i=1}^{n}\langle 2i-1\rangle!}$$

and combining with (9) gives (8). □

Theorem 12. *Let λ be a partition with at most n parts. Then*

$$o_\lambda(q^2,q^4,q^6,\dots,q^{2n}) = \prod_{(i,j)\in[\lambda]}\frac{\langle 2n+1+r'_\lambda(i,j)\rangle}{\langle h(i,j)\rangle}.$$

Let $|T|$ be the sum of the entries of the odd orthogonal tableau T where $\bar{\imath}$ is counted as $-i$ and ∞ is omitted. In this case,

$$o_\lambda(q^2,q^4,\dots,q^{2n}) = \sum_T q^{2|T|},$$

where T runs over the odd orthogonal λ-tableaux, and Theorem 12 gives a generating function for these tableaux. Again, setting $q=1$ yields King and El-Samra's expression [5] for the dimension of the irreducible polynomial $\mathrm{O}(2n+1)$-module with highest weight λ.

Corollary 13. *The number of semistandard odd orthogonal λ-tableaux with entries in the set $\{1,\overline{1},2,\overline{2},\dots,n,\overline{n},\infty\}$ is*

$$d_o(\lambda,2n+1) = \prod_{(i,j)\in[\lambda]}\frac{2n+1+r'_\lambda(i,j)}{h(i,j)}.$$

6. Generating function for semistandard even orthogonal tableaux

Finally, we consider the even orthogonal tableaux. Here there are two cases for the determinantal formula (see [4, pp. 410–411] or [10, p. 356]): if λ has strictly fewer than n parts then

$$o_\lambda(x_1,\ldots,x_n) = \frac{|x_j^{\lambda_i+n-i} + x_j^{-\lambda_i-n+i}|_{i,j=1}^n}{|x_j^{n-i} + x_j^{-n+i}|_{i,j=1}^n}$$

whereas for λ with exactly n parts

$$o_\lambda(x_1,\ldots,x_n) = 2\frac{|x_j^{\lambda_i+n-i} + x_j^{-\lambda_i-n+i}|_{i,j=1}^n}{|x_j^{n-i} + x_j^{-n+i}|_{i,j=1}^n}. \tag{10}$$

Lemma 14. *Let λ be a partition with at most n parts and set $\mu_i = \lambda_i + n - i$. Then*

$$o_\lambda(q, q^3, q^5, \ldots, q^{2n-1}) = \frac{\prod_{i=1}^n \langle 2\mu_i\rangle \prod_{1\le i<j\le n}\langle \mu_i-\mu_j\rangle\langle \mu_i+\mu_j\rangle}{\prod_{i=1}^n\langle \mu_i\rangle \prod_{i=1}^{n-1}\langle 2i\rangle!}.$$

Proof. Setting $x_j = q^{2j-1}$ we note that

$$d = |q^{(2j-1)(n-i)} + q^{-(2j-1)(n-i)}|_{i,j=1}^n = 2(-1)^{n(n-1)/2}|(q^{(2j-1)} + q^{-(2j-1)})^{i-1}|_{i,j=1}^n,$$

since we need to factor 2 out of the last row. So, by the argument in Lemma 4,

$$d = 2(-1)^{n(n-1)/2}\prod_{1\le i<j\le n}\langle i+j-1\rangle\langle j-i\rangle = 2(-1)^{n(n-1)/2}\prod_{i=1}^{n-1}\langle 2i\rangle!.$$

Further,

$$d' = |q^{(2j-1)\mu_i} + q^{-(2j-1)\mu_i}|_{i,j=1}^n = \prod_{i=1}^n (q^{\mu_i}+q^{-\mu_i})|(q^{\mu_i}+q^{-\mu_i})^{2(j-1)}|_{i,j=1}^n$$

where

$$|(q^{\mu_i}+q^{-\mu_i})^{2(j-1)}|_{i,j=1}^n = (-1)^{n(n-1)/2}\prod_{1\le i<j\le n}(q^{\mu_j-\mu_i} - q^{\mu_i-\mu_j})(q^{\mu_i+\mu_j} - q^{-\mu_i-\mu_j}).$$

When λ has exactly n parts we may replace $q^{\mu_i}+q^{-\mu_i}$ by $\langle 2\mu_i\rangle/\langle\mu_i\rangle$ for each i. This means that

$$d' = (-1)^{n(n-1)/2}\prod_{i=1}^n\frac{\langle 2\mu_i\rangle}{\langle\mu_i\rangle}\prod_{1\le i<j\le n}\langle\mu_j-\mu_i\rangle\langle\mu_i+\mu_j\rangle$$

and the result holds since the additional factor of 2 in (10) cancels with the denominator. However, when λ has fewer than n parts we have $\mu_n = 0$ so $q^{\mu_n}+q^{-\mu_n} = 2$ while $\langle 2\mu_n\rangle/\langle\mu_n\rangle = 1$. In this case,

$$d' = 2(-1)^{n(n-1)/2}\prod_{i=1}^n\frac{\langle 2\mu_i\rangle}{\langle\mu_i\rangle}\prod_{1\le i<j\le n}\langle\mu_j-\mu_i\rangle\langle\mu_i+\mu_j\rangle$$

and the result again holds. $\square$

The content $r'_\lambda(i,j)$ of the (i,j)th box in an even orthogonal tableaux is the same as that for the odd orthogonal tableaux. The following result can thus be deduced from Lemma 11 by subtracting 1 from each term and writing $\langle \mu_i - 1\rangle! = \langle \mu_i\rangle!/\langle \mu_i\rangle$.

Lemma 15. *Let λ be a partition with at most n parts. Then*

$$\prod_{(i,j)\in[\lambda]} \langle 2n + r'_\lambda(i,j)\rangle = \frac{\prod_{i=1}^{n}\langle \mu_i\rangle! \prod_{1\le i\le j\le n}\langle \mu_i+\mu_j\rangle}{\prod_{i=1}^{n}\langle \mu_i\rangle \prod_{i=1}^{n-1}\langle 2i\rangle!}.$$

Theorem 16. *Let λ be a partition with at most n parts. Then*

$$o_\lambda(q,q^3,q^5,\dots,q^{2n-1}) = \prod_{(i,j)\in[\lambda]} \frac{\langle 2n + r'_\lambda(i,j)\rangle}{\langle h(i,j)\rangle}.$$

As before, let $|T|$ denote the sum of the entries of an even orthogonal tableaux T and let $r(T) = r_+(T) - r_-(T)$ where $r_+(T)$ and $r_-(T)$ are the number of boxes in T containing symbols from the sets $\{1,2,\dots,n\}$ and $\{\overline{1},\overline{2},\dots,\overline{n}\}$, respectively. Then

$$o_\lambda(q,q^3,q^5,\dots,q^{2n-1}) = \sum_T q^{2|T|-r(T)},$$

where the sum runs over the even orthogonal λ-tableaux.

Corollary 17. *The number of semistandard even orthogonal λ-tableaux with entries in the set $\{1,\overline{1},2,\overline{2},\dots,n,\overline{n}\}$ is*

$$d_o(\lambda,2n) = \prod_{(i,j)\in[\lambda]} \frac{2n + r'_\lambda(i,j)}{h(i,j)}.$$

Remark 18. Suppose that λ has exactly n parts. Then the irreducible SO$(2n)$-modules of highest weights $\lambda^+ = (\lambda_1,\dots,\lambda_n)$ and $\lambda^- = (\lambda_1,\dots,-\lambda_n)$ have characters

$$so_{\lambda^\pm}(x_1,\dots,x_n) = \frac{|x_j^{\lambda_i+n-i} + x_j^{-\lambda_i-n+i}|_{i,j=1}^{n} \pm |x_j^{\lambda_i+n-i} - x_j^{-\lambda_i-n+i}|_{i,j=1}^{n}}{|x_j^{n-i} + x_j^{-n+i}|_{i,j=1}^{n}}.$$

Here it is more convenient to use the specialisation $x_j = q^{2(j-1)}$ since this eliminates the second term in the numerator above and we obtain

$$so_{\lambda^\pm}(1,q^2,q^5,\dots,q^{2n-2}) = \frac{\prod_{1\le i<j\le n}\langle \mu_i-\mu_j\rangle\langle \mu_i+\mu_j\rangle}{\prod_{i=1}^{n-1}\langle 2i-1\rangle!\langle i\rangle}.$$

Thus in both cases we have

$$so_{\lambda^\pm}(1,q^2,q^5,\dots,q^{2n-2}) = \prod_{i=1}^{n-1}\frac{\langle 2i\rangle}{\langle i\rangle}\prod_{i=1}^{n}\frac{\langle \mu_i\rangle}{\langle 2\mu_i\rangle}\prod_{(i,j)\in[\lambda]}\frac{\langle 2n + r'_\lambda(i,j)\rangle}{\langle h(i,j)\rangle}$$

since $\mu_n > 0$. It is clear that when we set $q = 1$ the additional terms reduce to $1/2$ so we find that the number of positive or negative semistandard even orthogonal λ-tableaux with entries in the set $\{1,\overline{1},2,\overline{2},\dots,n,\overline{n}\}$ is

$$d_{so}(\lambda^\pm,2n) = \frac{1}{2}\prod_{(i,j)\in[\lambda]}\frac{2n + r'_\lambda(i,j)}{h(i,j)}.$$

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